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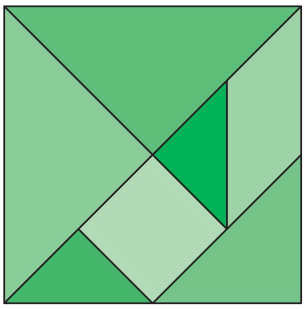
Tangrams and constraint-based geometry

Legend has it that a servant of a Chinese Emperor was carrying a very expensive square ceramic tray and when he tripped and fell it was shattered into seven pieces (called tans). He was not able to arrange the pieces back into the proper shape but he did realise that there were many other shapes that could be built from the pieces



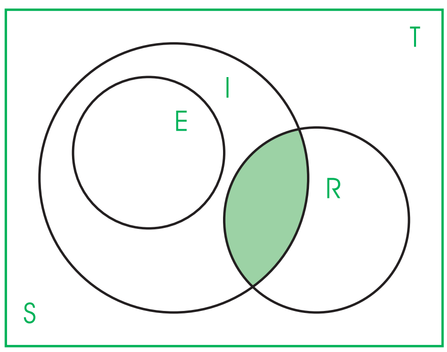
Tangrams have sometimes been used as an extension activity intended only to keep faster students busy while others finished essential desk-work. Without adequate introduction, many find that tangrams are just an open-ended form of jigsaw puzzle. Happily teachers have discovered that games provide an effective introduction to a new topic. In the case of tangrams, students are likely to learn more from their construction than from playing with the finished product.

This article describes how students can construct tangrams within a constraint-based geometric (CBG) environment, thereby learning much more than might be gained using scissors and cardboard, while at the same time learning to use the CBG system. The examples used here were constructed using a ClassPad 300 but might just as easily be developed using the Windows-based software package *Geometry Expressions*.

A major hurdle is one of definition. We may not expect too much trouble with student recognition of the parallelogram, square and three different sizes of right-angled isosceles triangles. However, if we revise these shapes within the context of the rest of the triangles and quadrilaterals, there are some very strange ideas out there. While looking for suitable Christmas presents for my grand-children I learned that a kite has a rhombus shape and a picture of a rhombus was described as a diamond.

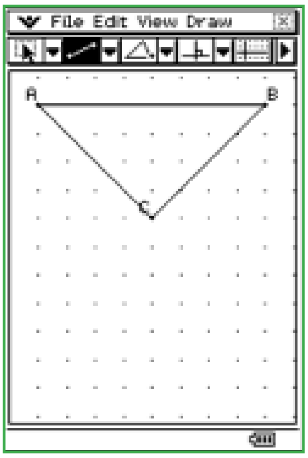
This is only part of the price we have paid for neglecting a generation. Almost 2000 years after Euclid systematised geometry we now have some teachers who have never been taught Euclidean Geometry. We should not be surprised if such folk think that geometry definitions are arbitrary—a matter of taste. A new textbook claims that an isosceles triangle has *only* two sides equal. The glossary in the back of the book contradicts the text and the worked examples correctly assume that an isosceles triangle has *at least* two sides equal. The lack of logic and consistency remains undetected by both author and proof-readers. We have not only discarded a knowledge system but we have failed to adopt an alternative geometry within which to teach the logical thinking Euclid once inspired.

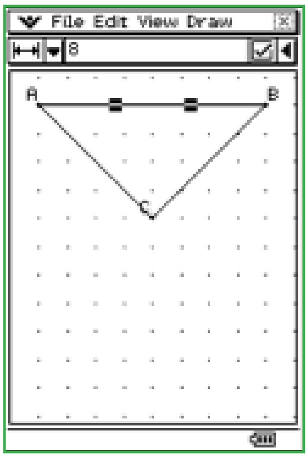
Many mathematics teachers are familiar with the Venn Diagram classification of triangles that shows clearly that the set of equilateral triangles (E) is a proper sub-set of the isosceles triangles (I). In the case of tangrams we are specifically interested in those triangles which are both isosceles (I) and right-angled (R).



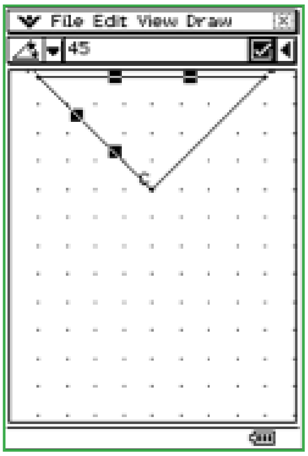
If students have learned the Theorem attributed to Pythagoras, this is an excellent opportunity to reconsider the 1:1:√2 triangle that Hippasus used to discover the existence of irrational numbers. Students love the story of how this led to his murder.

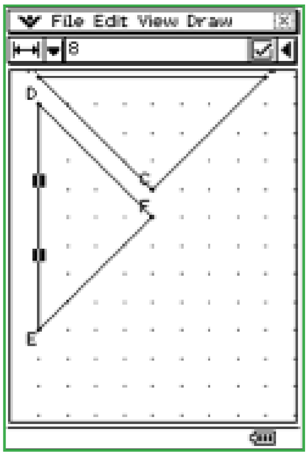
Open the Geometry application of a ClassPad and switch on the Integer Grid. Join three points, A, B and C as shown here.





Select the line segment AB and from the Measurement Bar fasten the length at 8 units by tapping the tick near the right hand end. If we attempt to move the triangle, the length AB will always have this fixed length of 8 units until such time as we remove the constraint on the length of AB. There are now several ways to fasten the size and shape of the whole triangle—what a wonderful opportunity to revise the conditions of congruency. For example, you could now select both AB and AC and from the Measurement Bar and constrain the angle BAC to 45°. If you do the same to angle ABC the size and shape of the triangle is fixed by the ASA condition. If you switch off the integer grid you will find that this first tan is easily rotated and moved.





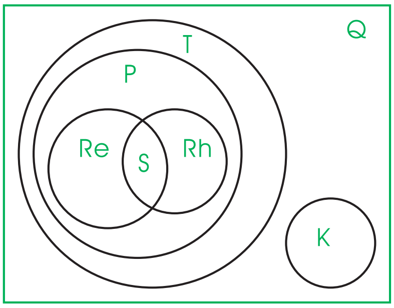
If you construct three more triangles, one for each condition of congruency, you can compare them for ease of manoeuvrability. For example, when I constrained the right-angle and the two equal sides (SAS), an attempt to fix the attitude of the triangle by constraining the slope of the hypotenuse was too much for the system. I found that the (ASA) tan was the most manoeuvrable in the ClassPad environment and so I used (ASA) for all of the triangles.

Triangles are particularly rigid shapes. Metal towers and cranes are made up of triangles. Quadrilaterals are not like that. They very easily flex to give a variety of shapes.

There is an important difference between dictionary style definitions that simply list the properties of a figure and the minimalist type definitions that can be used to solve problems in geometry. Here are the traditional quadrilateral definitions:

- a quadrilateral (Q) is a four-sided polygon.
- a parallelogram (P) is a quadrilateral with two pairs of opposite sides parallel.
- a rectangle (Re) is a parallelogram with at least one pair of sides perpendicular.
- a square (S) is a rectangle with at least one pair of adjacent sides equal.
- a rhombus (Rh) is a parallelogram with at least one pair of adjacent sides equal.

All the rest of the properties can be proved from this minimal amount of information. These definitions are usually supported by the following Venn Diagram.



Using these definitions, the necessary conditions for a parallelogram are:

- opposite sides parallel;
- opposite sides equal in length;
- opposite angles equal;
- diagonals bisect each other;
- one pair of sides equal and parallel.

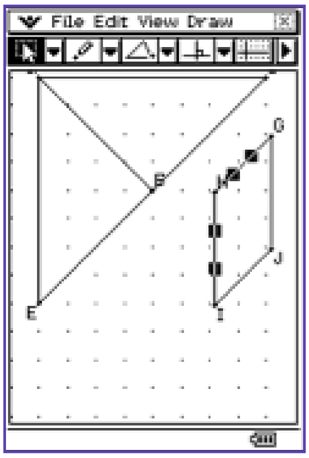
Even though these conditions determine that a quadrilateral is also a parallelogram, they do not fix its shape. Opposite angles may be equal but that does not determine the size. Opposite

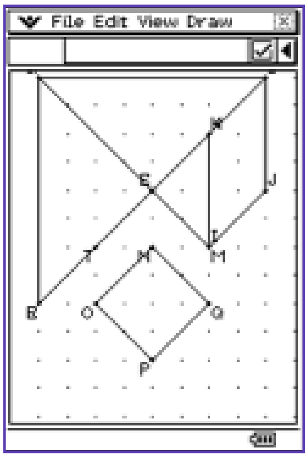
sides may be equal but that does not determine the shape. Suppose we switch on the integer grid and build a square and then constrain the length of every side. As soon as we switch the integer grid off, the square is free to slop around as a variable shaped rhombus.

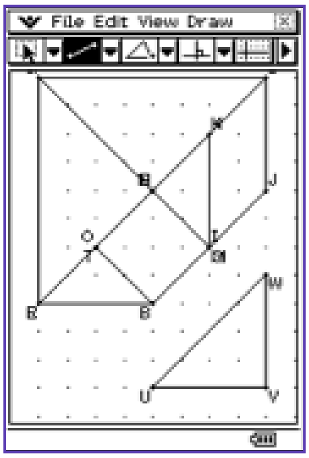
We can add one more constraint to stabilise the figure, such as fixing the length of a diagonal, but if we try to fix the length of both diagonals a CBG system will recognise that there is too much information and will ask which constraint you wish to remove so as to avoid contradictions. However, if you constrain only one diagonal the two free vertices can snap together forming a V-shape that does not contradict the constraints.

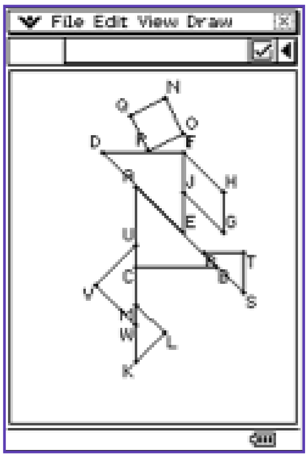
It is unwise to constrain the slopes of sides because this may make the shape more difficult to rotate. When later you come to build tangram pictures you will want to constrain some slopes and points after placement, so as to stabilise the growing picture. You may become confused if some of the figures have been defined using a constrained slope.

The strategy used here, for both the square and the parallelogram, is to constrain all the sides to the lengths defined by the grid and then to constrain one angle. In the case of the parallelogram, constrain the angle IHG (as shown below) because the resulting figure is less likely to be pulled out of shape. After finishing each shape, push it into place and save your work.



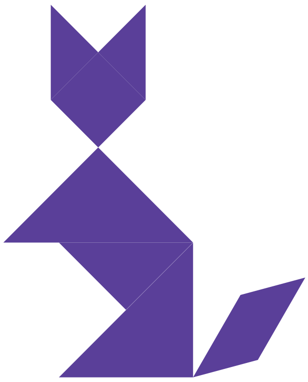


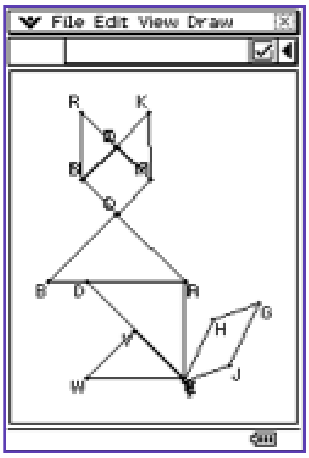




So we finally have a set of tangrams forming the original square pattern. Using a stylus each tan can be rotated or moved across the screen to form a variety of pictures and patterns. Some students will prefer to create their own shapes while more convergent thinkers may need outlines which they can attempt to match as shown below. You can obtain many extra examples from
http://www.tangrams.ca/Inner/down.htm
Use of material from this excellent site is free for non-profit use and education.

You will find that students learn much about the CBG environment by manipulating the tans to form different shapes. It is a more demanding process than the corresponding manipulation of cardboard pieces.





I found that the most successful technique is to rotate a tan by constraining one side to a particular slope and then to move the tan into position by constraining the distance between two matching points to zero.

This exercise introduces several key strategies when using CBG systems and prepares students for further use of this environment.

Editor's note: Years ago I found a delightful book titled *A Tangram Tale* that tells the story of the Willow pattern plate, using tangram-figures. I still use it with students to illustrate John Gough's point (in this issue) about using literature to teach mathematics.